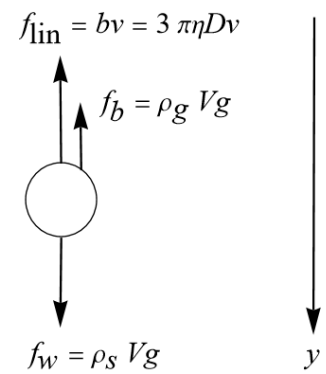
Problem 2.10

For a steel ball bearing (diameter 2 mm and density 7.8 g/cm^3) dropped in glycerin (density 1.3 g/cm^3 and viscosity 12 $\text{N}\cdot\text{s/m}^2$ at STP), the dominant drag force is the linear drag given by (2.82) of Problem 2.2. **(a)** Find the characteristic time τ and the terminal speed v_{ter}. [In finding the latter, you should include the buoyant force of Archimedes. This just adds a third force on the right side of Equation (2.25).] How long after it is dropped from rest will the ball bearing have reached 95% of its terminal speed? **(b)** Use (2.82) and (2.84) (with $\kappa = 1/4$ since the ball bearing is a sphere) to compute the ratio $f_{\text{quad}}/f_{\text{lin}}$ at the terminal speed. Was it a good approximation to neglect f_{quad}?

Solution

Part (a)

Draw a free body diagram for the steel ball falling in a medium with linear air resistance.



According to Stokes's law (Equation 2.82 on page 72), the drag is $f_{\text{lin}} = 3\pi\eta Dv$. The two other forces are the weight of the bearing and the buoyant force. The weight of the bearing is $f_w = mg = (\rho_s V)g$, where ρ_s is the density of steel. By Archimedes's principle, the buoyant force is equal to the weight of the fluid displaced by the ball, so $f_b = (\rho_g V)g$, where ρ_g is the density of glycerin. Apply Newton's second law in the y-direction, letting $v_y = v$.

$$\sum F_y = ma_y$$

$$\rho_s V g - 3\pi\eta D v - \rho_g V g = (\rho_s V)\frac{dv}{dt}$$

Bring the terms with v to one side of the equation.

$$\rho_s V \frac{dv}{dt} + 3\pi\eta D v = (\rho_s - \rho_g)Vg$$

Divide both sides by $\rho_s V$.

$$\frac{dv}{dt} + \frac{3\pi\eta D}{\rho_s V}v = \left(1 - \frac{\rho_g}{\rho_s}\right)g$$

Note that the volume of a ball is $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 = \frac{1}{6}\pi D^3$.

$$\frac{dv}{dt} + \frac{3\pi\eta D}{\rho_s\left(\frac{1}{6}\pi D^3\right)}v = \left(1 - \frac{\rho_g}{\rho_s}\right)g$$

Simplify the second term on the left.

$$\frac{dv}{dt} + \frac{18\eta}{\rho_s D^2}v = \left(1 - \frac{\rho_g}{\rho_s}\right)g$$

This is a first-order linear inhomogeneous ODE, so it can be solved by using an integrating factor.

$$I = \exp\left(\int^t \frac{18\eta}{\rho_s D^2}\,dt'\right) = \exp\left(\frac{18\eta}{\rho_s D^2}t\right)$$

Multiply both sides of the ODE by the integrating factor.

$$\exp\left(\frac{18\eta}{\rho_s D^2}t\right)\frac{dv}{dt} + \frac{18\eta}{\rho_s D^2}\exp\left(\frac{18\eta}{\rho_s D^2}t\right)v = \left(1 - \frac{\rho_g}{\rho_s}\right)g\exp\left(\frac{18\eta}{\rho_s D^2}t\right)$$

The left side can be written as $\frac{d}{dt}(Iv)$ by the product rule.

$$\frac{d}{dt}\left[\exp\left(\frac{18\eta}{\rho_s D^2}t\right)v\right] = \left(1 - \frac{\rho_g}{\rho_s}\right)g\exp\left(\frac{18\eta}{\rho_s D^2}t\right)$$

Integrate both sides with respect to t.

$$\exp\left(\frac{18\eta}{\rho_s D^2}t\right)v = \left(1 - \frac{\rho_g}{\rho_s}\right)g\frac{\rho_s D^2}{18\eta}\exp\left(\frac{18\eta}{\rho_s D^2}t\right) + C$$

Divide both sides by I.

$$v(t) = \left(1 - \frac{\rho_g}{\rho_s}\right)\frac{\rho_s g D^2}{18\eta} + C\exp\left(-\frac{18\eta}{\rho_s D^2}t\right)$$

Since the ball is dropped from rest, the initial velocity is zero: $v(0) = 0$. Use this fact to determine C.

$$v(0) = \left(1 - \frac{\rho_g}{\rho_s}\right)\frac{\rho_s g D^2}{18\eta} + C = 0 \quad \rightarrow \quad C = -\left(1 - \frac{\rho_g}{\rho_s}\right)\frac{\rho_s g D^2}{18\eta}$$

Therefore,

$$\begin{aligned} v(t) &= \left(1 - \frac{\rho_g}{\rho_s}\right)\frac{\rho_s g D^2}{18\eta} - \left(1 - \frac{\rho_g}{\rho_s}\right)\frac{\rho_s g D^2}{18\eta}\exp\left(-\frac{18\eta}{\rho_s D^2}t\right) \\ &= \left(1 - \frac{\rho_g}{\rho_s}\right)\frac{\rho_s g D^2}{18\eta}\left[1 - \exp\left(-\frac{18\eta}{\rho_s D^2}t\right)\right] \\ &= v_{\text{ter}}(1 - e^{-t/\tau}), \end{aligned}$$

where

$$\begin{aligned}
v_{\text{ter}} &= \left(1 - \frac{\rho_g}{\rho_s}\right) \frac{\rho_s g D^2}{18\eta} \\
&= \left(1 - \frac{1.3}{7.8}\right) \frac{\left[7.8\ \frac{\text{g}}{\text{cm}^3} \times \frac{1\ \text{kg}}{1000\ \text{g}} \times \left(\frac{100\ \text{cm}}{1\ \text{m}}\right)^3\right] \left(9.81\ \frac{\text{m}}{\text{s}^2}\right) \left(2\ \text{mm} \times \frac{1\ \text{m}}{1000\ \text{mm}}\right)^2}{18\left(12\ \frac{\text{N}\cdot\text{s}}{\text{m}^2}\right)} \\
&\approx 0.001\ \frac{\text{m}}{\text{s}} = 1\ \frac{\text{mm}}{\text{s}}
\end{aligned}$$

and

$$\begin{aligned}
\tau &= \frac{\rho_s D^2}{18\eta} \\
&= \frac{\left[7.8\ \frac{\text{g}}{\text{cm}^3} \times \frac{1\ \text{kg}}{1000\ \text{g}} \times \left(\frac{100\ \text{cm}}{1\ \text{m}}\right)^3\right] \left(2\ \text{mm} \times \frac{1\ \text{m}}{1000\ \text{mm}}\right)^2}{18\left(12\ \frac{\text{N}\cdot\text{s}}{\text{m}^2}\right)} \\
&\approx 0.0001\ \text{s} = 0.1\ \text{ms}.
\end{aligned}$$

In order to find how long it takes for the ball to reach 95% of its terminal speed, set $v(t) = 0.95 v_{\text{ter}}$ and solve the equation for t.

$$\begin{aligned}
v(t) &= v_{\text{ter}}(1 - e^{-t/\tau}) \\
0.95 v_{\text{ter}} &= v_{\text{ter}}(1 - e^{-t/\tau}) \\
0.95 &= 1 - e^{-t/\tau} \\
e^{-t/\tau} &= 0.05 \\
\ln e^{-t/\tau} &= \ln 0.05 \\
-\frac{t}{\tau} \ln e &= -\ln 20 \\
t &= \tau \ln 20 \\
&\approx 0.0004\ \text{s} = 0.4\ \text{ms}
\end{aligned}$$

Part (b)

The formula for f_{quad} is in Equation 2.84 on page 73, and the formula for f_{lin} is in Equation 2.82 on page 72. Calculate the ratio at the terminal velocity.

$$\begin{aligned}\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{\kappa\rho_g A v_{\text{ter}}^2}{3\pi\eta D v_{\text{ter}}} = \frac{\frac{1}{4}\rho_g\left(\frac{1}{4}\pi D^2\right)v_{\text{ter}}}{3\pi\eta D} &= \frac{\rho_g D v_{\text{ter}}}{48\eta} \\ &\approx \frac{\left[1.3\ \frac{\text{g}}{\text{cm}^3}\times\frac{1\text{ kg}}{1000\text{ g}}\times\left(\frac{100\text{ cm}}{1\text{ m}}\right)^3\right]\left(2\text{ mm}\times\frac{1\text{ m}}{1000\text{ mm}}\right)\left(0.001\ \frac{\text{m}}{\text{s}}\right)}{48\left(12\ \frac{\text{N}\cdot\text{s}}{\text{m}^2}\right)} \\ &\approx 5\times10^{-6}\end{aligned}$$

Since $f_{\text{quad}} \approx (5\times10^{-6})f_{\text{lin}}$, it's a very good approximation to neglect f_{quad}.